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Lower-dimensional pure-spinor superstrings

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Abstract: We study to what extent it is possible to generalise Berkovits' pure-spinor construction in $d = 10$ to lower dimensions. Using a suitable definition of a "pure" spinor in $d = 4, 6$, we propose models analogous to the $d = 10$ pure-spinor superstring in these dimensions. Similar models in $d = 2, 3$ are also briefly discussed.

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1. Introduction and summary

One of the most important developments in superstring theory in recent years is the discovery of the pure-spinor superstring [1]. In this new formulation, Lorentz covariance and supersymmetry are manifest and quantisation can be achieved. Among the virtues of the pure-spinor formalism are that it is possible to calculate scattering amplitudes, derive effective field theories and to compute higher-derivative corrections in a manifestly supersymmetric way, see e.g. [2, 3]. During the last few years the pure-spinor superstring has been studied and developed from several points of view, but further work is required in order to fully understand the basis of the model and its implications. For that reason, the study of similar but simpler models, based on the same structure, might shed some new light on the $d = 10$ formulation. In this note we construct and study some such models.

As is well known, at the classical level the superstring can be formulated in terms of different sigma models. One is the well-known RNS superstring. Another model, known in the literature as the Green-Schwarz (GS) superstring [4], has local worldsheet reparametrisation invariance and also a further symmetry, the so called κ-symmetry, which guarantees that target-space supersymmetry is implemented. However, no completely satisfactory way to quantise the GS model has been discovered to date. The only known way to quantise the GS superstring is in the light-cone gauge, but this approach has various drawbacks, e.g. Lorentz covariance is not manifest and the calculation of general scattering amplitudes is problematic.

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A way around the quantisation problems of the GS formulation is to add to the theory the variable [5] p_α, conjugate to the Grassmann-odd GS variable θ^α (here $\alpha = 1, \ldots, 16$) and in addition also add a Grassmann-even spinor ghost field [1] λ^α (and its conjugate w_α). In this way, the worldsheet action for the superstring in a flat ten-dimensional supergravity background becomes a quantisable free action with manifest spacetime supersymmetry. It turns out that only if one assumes that the ghost field λ^α is constrained can one make the total central charge vanish and make the Lorentz current algebra have the right properties. Berkovits discovered [1] that the appropriate constraint that needs to be imposed is $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$. The spinors λ^α satisfying this equation are called pure spinors.

The goal of this note is to investigate if it is possible to generalise Berkovits' construction for quantising the superstring in $d = 10$ to lower dimensions and, in particular, to those dimensions where (classically) a GS model exists, i.e. $d = (2), 3, 4, 6$. By analogy with the pure-spinor formalism in $d = 10$ we take the worldsheet ghost fields to involve a Grassmann-even spinor, λ, (and its conjugate momentum w) and look for suitable constraints on λ. We refer to the constrained spinor λ as a pure spinor in any dimension.

Even though a complete light-cone quantisation of the GS superstring is only possible in $d = 10$, in the particle limit the GS superstring becomes a superparticle theory [6] and one can quantise the theory in the light-cone gauge in all the dimensions listed above. In the particle limit, only the lowest state of the spectrum survives and one can check that it describes an on-shell massless gauge multiplet (for open superstrings) or an on-shell massless supergravity multiplet (for closed superstrings). Therefore, the counting of the degrees of freedom for the lowest multiplet is known and it can be checked if this counting matches the counting from a pure-spinor construction. In the next section, it will be shown how one can use this argument in reverse to obtain information about the required number of independent components of the pure spinor for each of the dimensions listed above. This gives a necessary condition on the pure spinor but it does not uniquely determine which constraint the pure spinor should satisfy.

Pure spinors (in the sense of Cartan) can be defined in any dimension, but that definition turns out to be not quite what we want. One clue about which constraint to use comes from the fact that in $d = 10$ the super-covariant derivatives satisfy $\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta} \partial_m$. In order for the BRST operator $Q_0 = \lambda^\alpha D_\alpha$ to be nilpotent one therefore requires $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$, i.e. the pure-spinor constraint. This construction is related to the general framework of spinorial cohomology proposed in [7, 8] and can be carried out in any dimension. In later sections it will be checked that if one defines the pure-spinor constraints in this way by starting from the algebra of super-covariant derivatives then, in any of the dimensions listed above, one finds that the pure spinor has the right number of independent components to ensure that the central charge is $c = 0$. Furthermore, one finds that the pure-spinor constraints obtained in this way are such that they imply $k = 1$ (Lorentz current algebra has level one) in all the dimensions listed above.

We should point out that ideas related to this work have appeared in various places in the literature, e.g. in section 2.6 of [9] where pure-spinor superparticle models were suggested. These models correspond to the particle limit of our models. Another related paper is [10] where more mathematical aspects were studied.

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This note is organised as follows. In the next section we first show how a heuristic counting of massless degrees of freedom gives the required number of independent components of the pure spinors in various dimensions. Then, using a decomposition of spinors similar to the one in [11], we show that the pure-spinor constraints obtained from the algebra of super-covariant derivatives lead to the same number of independent components as was obtained from the heuristic counting. In section 3 we present the worldsheet theories of the new pure-spinor models in $d = 3, 4, 6$. Then in section 4 we analyse the worldsheet conformal field theory of the pure-spinor models in $d = 4, 6$. In particular, we determine the central charge and calculate the current algebra of the Lorentz currents (more details of these calculations will be given in a separate publication [12]). In section 5, using a BRST operator analogous to the one proposed by Berkovits in $d = 10$, we analyse the cohomology at the massless level and show that the field content is exactly that of the super-Yang-Mills vector multiplet (further results will be presented in [13]). Finally, in section 6 we present our conclusions and discuss some open problems.

2. Pure-spinor constraints in various dimensions

In this section we present the pure-spinor constraints satisfied by the ghost field, λ, in our models. We first give a heuristic counting argument for how many independent components the pure spinor should possess. We then present the pure-spinor constraints and check that they are such that the number of independent components agree with the heuristic counting.

2.1 Counting of degrees of freedom

The number of independent degrees of freedom that the pure-spinor fields should possess can be investigated by a simple heuristic counting based on reparametrisation invariance and on the number of independent components of spinors in various dimensions.

If we denote by d_b the number of bosonic coordinates, by d_f the number of fermionic (Grassmann odd) coordinates, and by d_{ps} the number of independent components of the pure spinor. In order to have a "critical" (i.e. $c = 0$) model we need

$$d_b - 2d_f + 2d_{ps} = 0\,. \tag{2.1}$$

Using the known GS worldsheet structure (which gives d_b and d_f), we find that d_{ps} has to be 11 in $d = 10$, 5 in $d = 6$, and 2 in $d = 4$. For the $d = 3$ case, one finds the peculiar result $d_{ps} = 1/2$.

To implement the Virasoro constraints, we assume that two fermionic degrees of freedom have to be used to remove the lightcone coordinates $x^{\pm}$. We then find

$$(d_b - 2) - 2\,(d_f - d_{ps} - 1) = 0\,. \tag{2.2}$$

The same counting as in (2.2) also gives us the number of independent degrees of freedom for the lowest state of the spectrum. For example, using the GS formalism the number of fermionic fields after the κ-symmetry is taken into account is equal to $2\,(d_f - d_{ps} - 1)$. The above counting tells us that in $d = 10$ there are 8 bosonic massless degrees of freedom and 8 fermionic ones, in agreement with the (on-shell) spectrum of $d = 10$ super-Yang-Mills. In $d = 6$, the above counting gives 4 bosons plus 4 fermions, and so on. However, in lower dimensions knowledge of the number of independent degrees of freedom alone is not sufficient to unambiguously infer to which supermultiplet they belong.

2.2 The pure-spinor constraints

As discussed in the introduction, in $d = 10$ one can view the pure-spinor constraint as arising from the requirement that the BRST operator of the form $Q_0 = \lambda D$, where D is the super-covariant derivative, is nilpotent. Guided by this fact and the structure of the algebra of super-covariant derivatives in lower dimensions (see appendix A) we choose to impose the following "pure-spinor" constraints in $d = 4, 6$

$$\begin{aligned} \lambda\Gamma^m\lambda &= 0\,, \qquad (d=4) \\ \epsilon^{IJ}\lambda_I\gamma^m\lambda_J &= 0\,. \qquad (d=6) \end{aligned} \tag{2.3}$$

In $d = 4$ λ^α is a Dirac spinor and $\Gamma^m_{\alpha\beta}$ are the (symmetric) 4×4 gamma matrices, whereas in $d = 6$ λ^α_I $(I = 1, 2)$ is a doublet of Weyl spinors, and $\gamma^m_{\alpha\beta}$ are the (antisymmetric) 4×4 off-diagonal blocks ("Pauli matrices") in the Weyl representation of the 8×8 six-dimensional gamma matrices Γ^m. Note that the above $d = 6$ condition is not the conventional pure-spinor condition (which is solved by a Weyl spinor). However, as confusion is unlikely to arise we refer to the conditions (2.3) as pure-spinor conditions throughout this note. A convenient way to write the $d = 4$ pure-spinor condition is to use two-component Weyl notation: $\lambda^a\bar\lambda^{\dot a} = 0$ (here $a, \dot a = 1, 2$). In $d = 3$, a possible pure-spinor condition is given by the vector condition $(\alpha, \beta = 1, 2)$

$$\lambda\Gamma^m\lambda = 0 \quad \Leftrightarrow \quad \lambda^\alpha\lambda^\beta = 0\,, \qquad (d = 3)\,. \tag{2.4}$$

This condition has to be treated with care. Since the constraint is reducible it imposes less than 3 constraints on the spinor.

2.3 Solving the pure-spinor constraints

In $d = 10$, one can decompose the pure spinor as [11] $\lambda^\alpha = (S^a, S^{\dot a})$ where $a, \dot a = 1, \ldots, 8$. Using this decomposition, the pure-spinor constraint $\lambda\gamma^m\lambda = 0$ can be written

$$S^aS^b\delta_{ab} = 0\,, \qquad S^aS^{\dot b}\sigma^I_{a\dot b} = 0\,, \qquad S^{\dot a}S^{\dot b}\delta_{\dot a\dot b} = 0\,. \tag{2.5}$$

The first constraint reduces the number of independent components of S^a to 7. The other constraints are solved by an infinite number of gauge symmetries which reduce the number of components of $S^{\dot a}$ from 8 down to 4.[1]

[1] Let us also mention that in order to substantiate the argument given in subsection 2.1 on the use of reparametrisation invariance to count the degrees of freedom, one can relax the scalar constraint $S^aS^b\delta_{ab} = 0$ by adding it to the BRST operator. For consistency, one is then forced to also add a further piece to the BRST operator of the form $c\Pi$, where c is a Grassmann-odd ghost field and Π is the momentum along a specific direction. This new piece imposes a light-cone constraint on the coordinates x^m and plays the role of the Virasoro constraint.

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In the $d = 6$ case, one can use the decomposition $\lambda^\alpha_I = (S^a_I, S^{\dot a}_I)$ $(a, \dot a = 1, \dots, 2)$. The above pure-spinor constraint then decomposes into

$$S^a_I S^b_J \epsilon_{ab} \epsilon^{IJ} = 0\,, \qquad S^{\dot a}_I S^{\dot b}_J \epsilon_{\dot a \dot b} \epsilon^{IJ} = 0\,, \qquad S^a_I S^{\dot b}_J \epsilon^{IJ} = 0\,. \tag{2.6}$$

The second and third conditions are solved by introducing a new set of fields $\rho^{\dot b}_{(0)\,a}$ and by setting

$$S^{\dot b}_I = \rho^{\dot b}_{(0)\,a} S^a_I\,. \tag{2.7}$$

It can easily be checked that this solves the constraints above if the first constraint is satisfied. The new field $\rho^{\dot b}_{(0)\,a}$ has 4 degrees of freedom, but there are gauge symmetries to take into account. Indeed, we have that

$$\delta \rho^{\dot b}_{(n)\,a} = \epsilon^{IJ} \epsilon_{ab}\, \eta^{\dot b}_{(n)\,I} S^b_J\,, \qquad \delta \eta^{\dot b}_{(n)\,I} = \rho^{\dot b}_{(n+1)\,a} S^a_I\,, \qquad \dots \tag{2.8}$$

which leads to an infinite number of fields $\rho^{\dot \beta}_{(n)\,a}$ and $\eta^{\dot b}_{(n)\,I}$. This reduces the number of components of $S^{\dot a}_I$ from 4 to 2. The first constraint in (2.6) reduces the number of independent components of S^a_I down to 3. Hence, the total number of independent degrees of freedom is 5.

In the $d = 4$ case we use the decomposition $\lambda^\alpha = (S^\pm, \bar S^\pm)$. The pure-spinor constraint then decomposes into

$$S^+ \bar S^+ = 0\,, \qquad S^+ \bar S^- = 0\,, \qquad S^- \bar S^+ = 0\,, \qquad S^- \bar S^- = 0\,. \tag{2.9}$$

In analogy with the $d = 10$ and $d = 6$ cases, we solve the last 3 constraints by setting

$$S^- = \rho_{(0)} S^+\,, \qquad \bar S^- = \bar\rho_{(0)} \bar S^+\,, \tag{2.10}$$

which are defined up to the (infinite) gauge symmetries

$$\delta \rho_{(n)} = \eta_{(n)} S^+\,, \qquad \delta \bar\rho_{(n)} = \bar\eta_{(n)} \bar S^+\,, \qquad \delta \eta_{(n)} = \rho_{(n+1)} S^+\,, \qquad \delta \bar\eta_{(n)} = \bar\rho_{(n+1)} \bar S^+\,. \tag{2.11}$$

This reduces the pure spinors S^-, $\bar S^-$ to just one by means of an infinite number of fields. In addition, there is the first constraint $S^+ \bar S^+ = 0$, which also reduces the number of components by one, so the final solution is just 2 degrees of freedom.

We have shown that the number of independent components of the pure spinor is two in $d = 4$ and five in $d = 6$ (and eleven in $d = 10$). In other words, we see that in $d = 2n$ the pure-spinor condition eliminates n components from λ. An alternative way to obtain this result is to solve the pure-spinor constraints by temporarily breaking the SO($2n$) (Wick rotated) Lorentz invariance to U(n) $\simeq$ SU(n)$\times$U(1). This approach is discussed in more detail in [12].

We end this section by briefly discussing the $d = 3$ case. If we use the decomposition $\lambda^\alpha = (S^\pm)$, then the pure-spinor constraint becomes

$$S^+ S^+ = 0\,, \qquad S^+ S^- = 0\,, \qquad S^- S^- = 0\,. \tag{2.12}$$

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In analogy with the above situation, we solve the second and the last constraint by setting

$$S^- = \rho_{(0)} S^+ \,. \tag{2.13}$$

This is defined up to gauge symmetries given by

$$\delta\rho_{(n)} = \eta_{(n)} S^+ \,, \qquad \delta\eta_{(n)} = \rho_{(n+1)} S^+ \,. \tag{2.14}$$

This infinite gauge symmetry reduces the number of components of the "pure spinor" S^- to "1/2 degrees of freedom". It is not entirely clear how to interpret this result, but it will be shown in section 5 that the BRST cohomology makes sense and we explicitly compute the cohomology at the massless level. Further analysis and results will be presented elsewhere [13].

3. The $d = 3, 4, 6$ pure-spinor models

In this section we describe the worldsheet theories of the $d = 3, 4, 6$ pure-spinor models in a flat supergravity background. For simplicity we only write the left-moving sector explicitly.

For the case of $\mathcal{N} = 1$ supersymmetry in $d = 4$ we choose (following Siegel [5]) the left-moving (holomorphic) "matter" worldsheet fields to be $(x^m, \theta^\alpha, p_\alpha)$, where θ^α is a four-component Dirac spinor, and p_α is its conjugate momentum ($\alpha = 1, \ldots 4$). The Dirac spinor θ^α can be decomposed into a Weyl spinor, θ^a, and an anti-Weyl spinor, $\bar{\theta}^{\dot{a}}$. Similarly, p_α can be decomposed into p_a, and $\bar{p}_{\dot{a}}$.

For the case of $\mathcal{N} = (1, 0)$ supersymmetry in d=6 we choose the left-moving (holomorphic) matter worldsheet fields to be $(x^m, \theta^\alpha_I, p^I_\alpha)$, where θ^α_I is a doublet ($I = 1, 2$) of four-component Weyl spinors, and p^I_α are their conjugate momenta.

For the case of $\mathcal{N} = 1$ supersymmetry in $d = 3$ we choose the left-moving matter worldsheet fields to be $(x^m, \theta^\alpha, p_\alpha)$ where θ^α is a two-component Majorana spinor, and p_α is its conjugate momentum ($\alpha = 1, 2$). The Dirac gamma matrices $\Gamma^m_{\alpha\beta}$ are symmetric and any symmetric bispinor $A^{\alpha\beta}$ is proportional to the gamma matrix itself $A^{\alpha\beta} = \Gamma^{\alpha\beta}_m A^m$.

The worldsheet actions for the left-moving modes (in the conformal gauge) have the following form

$$\begin{aligned}
S^{d=3}_{\mathrm{left-moving}} &= \int d^2z \left(\eta_{mn}\partial x^m \bar{\partial} x^n + p_\alpha \bar{\partial}\theta^\alpha + w_\alpha \bar{\partial}\lambda^\alpha\right), \\
S^{d=4}_{\mathrm{left-moving}} &= \int d^2z \left(\eta_{mn}\partial x^m \bar{\partial} x^n + p_\alpha \bar{\partial}\theta^\alpha + w_\alpha \bar{\partial}\lambda^\alpha\right), \\
S^{d=6}_{\mathrm{left-moving}} &= \int d^2z \left(\eta_{mn}\partial x^m \bar{\partial} x^n + p^I_\alpha \bar{\partial}\theta^\alpha_I + w^I_\alpha \bar{\partial}\lambda^\alpha_I\right).
\end{aligned} \tag{3.1}$$

Note that the third term in each of these actions is written as a covariant expression, but one has to take into account that the pure spinors satisfy a constraint. The pure-spinor constraint implies that the action is invariant under the gauge symmetries

$$\begin{aligned}
\delta w_\alpha &= \Lambda_{\alpha\beta}\lambda^\beta \,, \qquad (d = 3) \\
\delta w_\alpha &= \Lambda_m \Gamma^m_{\alpha\beta}\lambda^\beta \,, \qquad (d = 4)
\end{aligned}$$

$$\delta w^I_\alpha = \Lambda_m \epsilon^{IJ} \gamma^m \lambda^\beta_J . \qquad (d=6) . \tag{3.2}$$

Some progress toward understanding these gauge symmetries has been made in [14] by adding new ghost fields. However, for the rest of the paper, we do not introduce any further degrees of freedom.

The only consistent way to study the BRST cohomology is to restrict the functional space consisting of pure spinors and their conjugates to the space consisting of only gauge invariant combinations. Using the gauge symmetries (3.2) one can easily check that the number of independent w_α and w^I_α's is the same as the number independent components of the pure spinors.

Notice that in order to be able to calculate correlation functions involving the pure-spinor fields using free field technology one has to fix the gauge to break the symmetry (3.2). This can be done in several ways, for instance by selecting a suitable representation of the pure spinors and choosing the gauge where the only non-vanishing components of w_α and w^I_α are exactly the variables conjugate to the independent components of the pure spinors. Computing correlation functions among gauge invariant operators, the details of the gauge fixings are irrelevant. An alternative method which does not rely on any particular gauge has been developed in [15].

It can easily be checked that the actions (3.1) are Lorentz invariant and invariant under supersymmetry. Notice that one of the main difference with GS superstrings is the fact that the actions can be written in terms of supersymmetric variables without introducing a Wess-Zumino term. Up to terms that vanish by the equations of motion, the supersymmetric variables are given by $\partial\theta^\alpha$ $(d=4)$, $\partial\theta^\alpha_I$ $(d=6)$ as well as

$$\begin{aligned} d_\alpha &= p_\alpha - \frac{1}{2}(\Gamma_m\theta)_\alpha \partial x^m - \frac{1}{8}(\Gamma_m\theta)_\alpha(\theta\Gamma^m\partial\theta) , \qquad (d=4) \\ d^I_\alpha &= p^I_\alpha - \frac{1}{2}\epsilon^{IJ}(\gamma_m\theta_J)_\alpha \partial x^m - \frac{1}{8}\epsilon^{IJ}(\gamma_m\theta_J)_\alpha \epsilon^{KL}(\theta_K\gamma^m\partial\theta_L) , \qquad (d=6) \end{aligned} \tag{3.3}$$

and also

$$\begin{aligned} \Pi^m &= \partial x^m + \frac{1}{2}\theta\Gamma^m\partial\theta , \qquad (d=4) \\ \Pi^m &= \partial x^m + \frac{1}{2}\epsilon^{IJ}(\theta_I\gamma^m\partial\theta_J) . \qquad (d=6) \end{aligned} \tag{3.4}$$

The $d=3$ supersymmetric variables can be constructed in an analogous way. However, since they will not be used in this paper, we do not present the explicit formulæ here.

4. The pure-spinor conformal field theory in $d=4,6$

In this section we discuss the worldsheet conformal field theory for the $d=4,6$ pure-spinor models. Additional details will appear in [12].

The free fields x^m, θ and p have the standard OPEs (in units where $\alpha'=2$),

$$x^m(y,\bar y)x^n(z,\bar z) \sim -\eta^{mn}\log|y-z|^2 , \qquad (d=4,6)$$

$$p_\alpha(y)\theta^\beta(z) \sim \frac{\delta_\alpha^\beta}{y-z}\,, \qquad (d=4)$$

$$p^I_\alpha(y)\theta^\beta_J(z) \sim \frac{\delta^I_J\delta_\alpha^\beta}{y-z}\,. \qquad (d=6) \tag{4.1}$$

In terms of the supersymmetric variables given in the previous section we have the following OPEs for $d=4$

$$\begin{aligned} d_\alpha(y)d_\beta(z) &\sim -\frac{1}{y-z}\Gamma^m_{\alpha\beta}\Pi_m(z)\,, \qquad d_\alpha(y)\Pi^m(z) \sim \frac{1}{y-z}\left(\Gamma^m\partial\theta\right)_\alpha(z)\,,\\ d_\alpha(y)\partial\theta^\beta(z) &\sim \frac{1}{(y-z)^2}\delta_\alpha^\beta\,, \end{aligned} \tag{4.2}$$

and for $d=6$

$$\begin{aligned} d^I_\alpha(y)d^J_\beta(z) &\sim -\frac{1}{y-z}\epsilon^{IJ}\gamma^m_{\alpha\beta}\Pi_m(z)\,, \qquad d^I_\alpha(y)\Pi^m(z) \sim \frac{1}{y-z}\epsilon^{IJ}\left(\gamma^m\partial\theta_I\right)_\alpha(z)\,,\\ d^I_\alpha(y)\partial\theta^\beta_J(z) &\sim \frac{1}{(y-z)^2}\delta^I_J\delta_\alpha^\beta\,. \end{aligned} \tag{4.3}$$

The “matter” part of the stress tensor is

$$\begin{aligned} T_{\rm mat} &= -\frac{1}{2}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha\,, \qquad (d=4)\\ T_{\rm mat} &= -\frac{1}{2}\partial x^m\partial x_m - p^I_\alpha\partial\theta^\alpha_I\,. \qquad (d=6) \end{aligned} \tag{4.4}$$

From these expressions we see that the x^m CFT has the standard $c=d$ central charge, while the (p,θ) CFT has central charge $c=-8$ ($d=4$) and $c=-16$ ($d=6$). For the total central charge to vanish, the ghost CFT has to have $c=4$ ($d=4$) and $c=10$ ($d=6$).

As in $d=10$, one can construct the manifestly SO$(2n)$ Lorentz-covariant quantities

$$\begin{aligned} N^{mn} &= \frac{1}{2}w\Gamma^{mn}\lambda\,, \qquad \partial h = \frac{1}{2}w\lambda\,, \qquad (d=4)\\ N^{mn} &= \frac{1}{2}w^I\gamma^{mn}\lambda_I\,, \qquad \partial h = \frac{1}{2}w^I\lambda_I\,. \qquad (d=6)\,. \end{aligned} \tag{4.5}$$

It can be shown [12] that, in SO$(2n)$ covariant notation, the OPEs involving N^{mn} and λ take the form

$$\begin{aligned} N^{mn}(y)\lambda^\alpha(z) &\sim \frac{1}{2}\,\frac{1}{y-z}\left(\Gamma^{mn}\lambda\right)^\alpha(z)\,, \qquad (d=4)\\ N^{mn}(y)\lambda^\alpha_I(z) &\sim \frac{1}{2}\,\frac{1}{y-z}(\gamma^{mn})^\alpha{}_\beta\,\lambda^\beta_I(z)\,, \qquad (d=6)\\ N^{pq}(y)N^{mn}(z) &\sim \frac{\eta^{pm}N^{qn}(z)-\eta^{qm}N^{pn}(z)-(m\leftrightarrow n)}{y-z} -\\ &\quad -(n-2)\frac{\eta^{pn}\eta^{qm}-\eta^{pm}\eta^{qn}}{(y-z)^2}\,. \end{aligned} \tag{4.6}$$

where n takes the values $2,3$ ($n=5$ corresponds to the $d=10$ case). From the last equation in (4.6) we see that the N^{mn}'s form an SO$(2n)$ current algebra with level $k=2-n$.

In $d = 4, 6$ (as in $d = 10$), ∂h has no singular OPEs with N^{mn} and satisfies

$$h(y)h(z) \sim -\log(y-z)\,, \qquad \partial h(y)\lambda(z) \sim \frac{1}{2}\,\frac{1}{y-z}\lambda(z)\,. \tag{4.7}$$

In comparison, the OPEs involving the (p,θ) Lorentz currents, $M^{mn} = -\frac{1}{2}p\Gamma^{mn}\theta$ ($d = 4$) and $M^{mn} = -\frac{1}{2}p^I\gamma^{mn}\theta_I$ ($d = 6$), take the form

$$M^{mn}(y)\theta^\alpha(z) \sim \frac{1}{2}\,\frac{1}{y-z}(\Gamma^{mn}\theta)^\alpha(z)\,, \qquad (d=4) \tag{4.8}$$

$$M^{mn}(y)\theta^\alpha_I(z) \sim \frac{1}{2}\,\frac{1}{y-z}(\gamma^{mn})^\alpha{}_\beta\,\theta^\beta_I(z)\,, \qquad (d=6) \tag{4.9}$$

$$M^{pq}(y)M^{mn}(z) \sim \frac{\eta^{pm}M^{qn}(z) - \eta^{qm}M^{pn}(z) - (m \leftrightarrow n)}{y-z} + (n-1)\frac{\eta^{pn}\eta^{qm} - \eta^{pm}\eta^{qn}}{(y-z)^2}\,.$$

Thus the M^{mn}'s form an $\mathrm{SO}(2n)$ current algebra with level $k = n - 1$. By combining the above results one finds that the total Lorentz current $L^{mn} = M^{mn} + N^{mn}$ satisfies the OPE

$$L^{pq}(y)L^{mn}(z) \sim \frac{\eta^{pm}L^{qn}(z) - \eta^{qm}L^{pn}(z) - (m \leftrightarrow n)}{y-z} + \frac{\eta^{pn}\eta^{qm} - \eta^{pm}\eta^{qn}}{(y-z)^2}\,, \tag{4.10}$$

and thus forms an $\mathrm{SO}(2n)$ current algebra with level $k = 1$ for all n, i.e. independently of the dimension.

Using the covariant fields N^{mn} and ∂h, *part of* the ghost stress tensor in $d = 4, 6, 10$ can be written as [12]

$$T_{N,\partial h} = -\frac{1}{4n}N_{mn}N^{mn} - \frac{1}{2}(\partial h)^2 + \frac{(n-1)}{2}\partial^2 h\,. \tag{4.11}$$

Above we said 'part of' since in $d = 6$ the stress tensor also contains the additional decoupled piece [12]

$$T_{u,v} = \partial u \partial v - \partial v^2\,, \tag{4.12}$$

where ∂u and ∂v satisfy the OPE $\partial u(y)\,\partial v(z) \sim (y-z)^{-2}$.

Let us analyse the ghost stress tensor (4.11) in more detail. The first piece involves the ghost Lorentz currents, N^{mn}, and is a Sugawara construction for an $\mathrm{SO}(2n)$ WZNW model with level $k = 2 - n$. Indeed, recalling that the dual Coxeter number of $\mathrm{SO}(2n)$ is $g^\vee = 2n - 2$, we find[2] $2(g^\vee + k) = 2n$. Using standard formulæ one finds that the central charge for the piece involving the ghost Lorentz currents is

$$c = \frac{k \dim \mathrm{SO}(2n)}{k + g^\vee} = (2-n)(2n-1)\,. \tag{4.13}$$

In (4.11) the second piece refers to a Coulomb gas, with a background charge of $Q = (n-1)$, and consequently central charge $c = 1 + 3Q^2 = 1 + 3(n-1)^2$. Adding these two contributions gives $n^2 - n + 2$. Taking into account that in $d = 6$ the additional piece (4.12) has $c = 2$

[2] Due to our normalisation of the NN OPE the prefactor in front of $N_{mn}N^{mn}$ in (4.11) is $-\frac{1}{4(k+g^\vee)}$. To obtain the usual $+\frac{1}{2(k+g^\vee)}$ one would have to rescale the currents N^{mn}.

we find that the central charge contribution from the ghost sector is $c = 4$ ($d = 4$), $c = 10$ ($d = 6$) and $c = 22$ ($d = 10$) and therefore the total central charge vanishes.

A comment is in order here. Since we have shown that the central charge vanishes and that the Lorentz current algebra closes, are we to interpret the models as critical superstrings in $d < 10$? We do not believe this to be the correct interpretation. Rather, the point of view that is taken in [12] is that the models should be thought of as the compactification-independent sector of the superstring compactified on a Calabi-Yau manifold CY_l down to $d = 10 - 2l$.

The zero-mode saturation rules that follow from the above analysis takes the schematic form (see [12] for more details)

$$\langle 0|\lambda^{n-2}\theta^n|\Omega\rangle \neq 0\,, \tag{4.14}$$

where (schematically) $|\Omega\rangle = \prod_A^q Y^A|0\rangle$ with (as in [2]) $Y^A = C^A\theta\delta(C^A\lambda)$. The integer q in this expression is given by 2 ($d = 4$), 5 ($d = 6$) and 11 ($d = 10$).

5. BRST operator, cohomology and vertex operators

In this section we initiate a discussion of the BRST structure of the lower-dimensional pure-spinor models. For simplicity, we only consider the open (or left-moving) sector of the superstring.

In the $d = 10$ pure-spinor formalism, the physical states of the superstring are obtained from vertex operators in the cohomology of the BRST operator (we only display the left-moving part)

$$Q = \oint \lambda^\alpha d_\alpha\,. \tag{5.1}$$

The natural guesses for the BRST operators of the lower-dimensional models are

$$\begin{aligned} Q &= \oint \lambda^\alpha d_\alpha\,, \qquad (d = 4) \\ Q &= \oint \lambda_I^\alpha d_\alpha^I\,, \qquad (d = 6)\,. \end{aligned} \tag{5.2}$$

These operators are nilpotent because of the pure-spinor constraints. Notice that a similar operator was used in [16]. There, the dimensional reduction to $d = 4$ preserving $\mathcal{N} = 2, 3, 4$ supersymmetry was studied and the relation with harmonic superspace was shown. One of the key points is that the BRST operator obtained after the introduction of harmonic variables becomes similar to the above operator and only after the introduction of additional constraints (analytical constraints) are the equations motions recovered. We now analyse the cohomology of the Q's given in (5.2) for massless states.[3]

In $d = 4$ at ghost number zero BRST-closedness requires $\lambda^\alpha D_\alpha\Phi = 0$. If this is to hold for *all* ways of solving the pure-spinor constraint then $D_\alpha\Phi = 0$ has to hold; thus $\Phi = \text{const}$ is the only solution. At ghost number one, if one works in the Weyl basis, one

[3]The solutions to the BRST cohomology equations can also be computed using the technique of spinorial cohomology discussed in [7, 8]. In addition, we should mention that there is vast literature on solving superspace constraints (Bianchi identities) in $d = 4$ and $d = 6$.

finds that in order for Q to give zero when acting on $\lambda^a A_a(x,\theta) + \bar\lambda^{\dot a}\bar A_{\dot a}(x,\theta)$, one has to have $A_a = D_a M - D_a N$ and $\bar A_{\dot a} = \bar D_{\dot a} M + \bar D_{\dot a} N$. Here the terms involving M correspond to a BRST exact piece, whereas the field content of N constitute the cohomology and is exactly that of (off-shell) $\mathcal{N}=1$, $d=4$ Yang-Mills. Note that Q does *not* put the theory on-shell; it only selects the right (off-shell) field content. To put the theory on-shell one needs to add the Virasoro constraint (and possibly some other constraints as well; superspace might give a clue about the minimal number of constraints one needs). We do not have a detailed understanding of how this works. In the Dirac basis, at ghost number one, we can write

$$U = \lambda^\alpha A_\alpha(x,\theta)\,. \tag{5.3}$$

The BRST condition $QU=0$ implies $\lambda^\alpha\lambda^\beta D_\lambda A_\beta = 0$. Therefore, the most general solution is given by

$$U = \lambda^\alpha\left(D_\alpha M + (\Gamma^{mnpq})_\alpha{}^\beta D_\beta N_{mnpq}\right) = \lambda^\alpha\left(D_\alpha M + (\Gamma^5)_\alpha{}^\beta D_\beta N_5\right). \tag{5.4}$$

The above vertex operator (5.3) is invariant under the gauge symmetry $\delta U = \{Q,\Omega\}$ for a scalar superfield Ω. Therefore the first term in (5.4) can be removed by a gauge transformation, and the second term represents the cohomology. Notice that we have to require the reality condition in order not to spoil the hermiticity of the theory.[4] This implies that the degrees of freedom are represented by a real scalar superfield N_5. At higher ghost number the cohomology is empty. This can be understood by computing the zero-momentum cohomology. Zero-momentum cohomology is the BRST cohomology computed on the space of functionals independent of the spacetime coordinates x^m. On general grounds it turns out that at any ghost number, the zero-momentum cohomology is larger than the non-zero one (only a subset of the cohomology classes of zero-momentum cohomology can be lifted to the non-zero momentum one) and, therefore, if the zero-momentum cohomology is empty at a given ghost number, also the corresponding non-zero momentum cohomology is empty. In the present case, the zero-momentum cohomology is given by the following generators

$$1\,,\quad \lambda^a\bar\theta^{\dot a} + \bar\lambda^{\dot a}\theta^a\,,\qquad \lambda^a\bar\theta^2 - \bar\lambda^{\dot a}\bar\theta_{\dot a}\theta^a\,,\qquad \bar\lambda^{\dot a}\theta^2 - \lambda^a\theta_a\bar\theta^{\dot a}\,,\qquad \bar\lambda^{\dot a}\bar\theta_{\dot a}\theta^2 - \lambda^a\theta_a\bar\theta^2\,. \tag{5.5}$$

One can also analyse the cohomology at the first massive level. This analysis is performed in [13].

Integrated vertex operators can be constructed in same way as in $d=10$, i.e. via the descent equation: if $\mathcal{V}$ is the integrand of the integrated vertex operator then (for open

[4] A remark is in order here. In $d=10$ and in lower dimensions, the solution of pure-spinor constraints can be only achieved by using complex spinorial fields. For example in $d=10$, the pure spinors are only Weyl and not Majorana-Weyl. However, at the level of the path integral and in all manipulations only the field λ appears whereas its complex conjugate is absent. However, if the pure-spinor constraints are not solved explicitly, one does not see this phenomenon. In $d=4$, the explicit solution of the pure spinor constraint breaks the hermiticity properties, but it does not spoil the Lorentz invariance of the theory.

superstrings) $[Q, \mathcal{V}] = \partial U$. The massless integrated vertex operator can be written in the Weyl basis as

$$V = \oint \left[\partial x^m A_m + \partial\theta^a A_a + \partial\bar{\theta}^{\dot{a}} \bar{A}_{\dot{a}} + d_a W^a + \bar{d}_{\dot{a}} \bar{W}^{\dot{a}} + \frac{1}{2} N^{mn} F_{mn} \right], \tag{5.6}$$

where the decent equation implies that the superfields A_m, W^a, $\bar{W}^{\dot{a}}$ and F_{mn} satisfy the usual $d = 4$ SYM superspace relations, and A_a and $\bar{A}_{\dot{a}}$ are as above. Plugging these results into the above expression (5.6) one finds exact agreement with the vertex operator in the hybrid superstring [17], except for the fact that the term involving N^{mn} is not present in the hybrid superstring. This is discussed further in [12].

We have not studied the construction of scattering amplitudes using the above vertex operators, but we note that from the saturation rule given in (4.14) and the ghost number of U it seems hard to use the same prescription as in $d = 10$ [1].

This concludes our discussion of the massless cohomology in $d = 4$. But before we move on to the $d = 6$ case a comment is in order. There is also another way one could compute the cohomology: first solve the pure-spinor constraint and plug the solution into Q, and then compute the BRST cohomology. This is rather easy to analyse in the present case. As we saw in section 2, the pure-spinor constraint is solved by choosing either $\lambda^a = 0$ or $\lambda^{\dot{a}} = 0$ (at the level of the path integral both solutions have to be taken into account; this will be discussed in a forthcoming paper [13]). By plugging e.g. the solution $\lambda^a = 0$ into the BRST formula, one finds the new operator $Q' = \oint \lambda^{\dot{a}} d_{\dot{a}}$ which of course is also nilpotent. The massless cohomology of this new operator can be straightforwardly computed: every chiral superfield $\Phi(y, \theta)$ satisfying $D_{\dot{a}}\Phi = 0$ is an element of the cohomology at ghost number zero. A chiral superfield has 4 bosonic and 4 fermionic degrees of freedom. It has the same number of independent (off-shell) components as the gauge multiplet which occurred in the calculation above and indeed they can be mapped into each other. (A similar phenomenon is discussed in [18].) The exact relation between the two cohomologies needs to be understood better.

In $d = 6$ the cohomology, at zero ghost number, of the operator in (5.2) again contains only a constant. At ghost-number one the unintegrated vertex operator can be written $U = \lambda^\alpha_I A^I_\alpha$. The BRST condition implies $D^{(I}_{(\alpha} A^{J)}_{\beta)} = 0$ which is solved by [19]

$$A^J_\alpha = D^J_\alpha M + \epsilon_{KI} D^I_\alpha D^{JKLM} N_{LM}\,, \tag{5.7}$$

where $D^{JKLM} = \epsilon^{\alpha\beta\gamma\delta} D^{(J}_\alpha D^K_\beta D^L_\gamma D^{M)}_\delta$. Here the term involving M is BRST exact, whereas it is known [19] that N_{LM} describes $\mathcal{N} = 1$ $d = 6$ (off-shell) super-Yang-Mills [19]. The $d = 6$ cohomology was also computed in [8] using the (equivalent) spinorial cohomology framework. As in $d = 4$ one can also investigate integrated vertex operators etc. but we will not do so here.

Despite the strange nature of the pure spinors in $d = 3$ this case can also be analysed using the above method. It will now be shown that the cohomology is well defined and at the lowest (massless) level describes the gauge supermultiplet.

The most general vertex operator of ghost number 1 at the massless level is

$$U = \lambda^\alpha A_\alpha(x,\theta)\,, \tag{5.8}$$

which is defined up to the gauge transformation $\delta A_\alpha = D_\alpha \Omega$ for some scalar superfield Ω. Imposing BRST invariance and using the pure-spinor condition, we see that there is no constraint on the superfield A_α. However, by using the gauge symmetry, we can easily see that

$$A_\alpha = a_{(\alpha\beta)}(x)\theta^\beta + u_\alpha(x)\theta^2\,, \qquad \delta a_{\alpha\beta}(x) = \partial_{\alpha\beta}\omega(x)\,. \tag{5.9}$$

This is exactly the field content and gauge symmetry of super-Yang-Mills in $d=3$. It is easy to analyse also the higher ghost-number cohomology and it turns out that there no massless cohomology with ghost number larger than 1. This poses the problem how to construct antifields, but this question is also related to the prescription for the saturation rule which is not discussed in this note. The computation of the cohomology at the massive levels is left to a separate publication [13].

An even simpler (albeit somewhat degenerate) case occurs in $d=2$. Consider the case of $\mathcal{N}=(2,0)$ supersymmetry with superspace variables $(x^m, \theta^\alpha_I, p^I_\alpha)$, where $I=1,2$ labels a doublet of Majorana-Weyl spinors. We take the pure-spinor constraint satisfied by λ_I to be

$$\delta^{IJ}\lambda_I\gamma^m\lambda_J = 0\,, \qquad I,J = 1,2\,. \tag{5.10}$$

As in $d=3,4,6$ the only cohomology at ghost number zero is a constant. At ghost number one, acting with Q on $\lambda_I A^I$ one finds the constraint $D^{(I}A^{J)} - \frac{1}{2}\delta^{IJ}D^K A^L \delta_{KL} = 0$. The solution is $A_I = D_I M - \epsilon_{IJ}D_J N$, or $A_1 = D_1 M - D_2 N$ and $A_2 = D_2 M + D_1 N$. Here the piece involving M is BRST-trivial, whereas the cohomology is contained in N. More details about this $d=2$ pure-spinor model are presented in [12].

6. Open problems and discussion

In this note we have summarised some basic facts about pure-spinor superstrings in lower dimensions. A lot of work can be done and, in our opinion, should be done in this direction. Some immediate questions which arise are how/if these models are related to compactifications of the RNS superstring, and how/if they are related to the so called hybrid superstrings [17, 20] which describe compactifications of the superstring with manifest super-Poincaré covariance in the non-compact directions. Some of these questions are touched upon in [12].

The new models are simpler than the $d=10$ pure-spinor superstring and provide a nice laboratory for studying quantum field theories with constraints. One of the most interesting aspects which is not yet understood (even for the $d=10$ model) is the origin of the BRST operator.[5]

[5] We should mention that it has been argued [21] that the pure-spinor formulation can be obtained from the superembedding formalism. In addition, in the modified GS action given in [22], one can see the pure-spinor BRST operator emerging. Finally, in [23] the BFT embedding is used to motivate the BRST operator.

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Another interesting question to understand is how the Virasoro constraints are implemented in the new framework. Let us elaborate on this issue. A basic ingredient of string theory is the conformal invariance and the reparametrisation invariance on the worldsheet. At tree level a detailed understanding is perhaps not really needed, but at higher genus the conformal invariance and the Virasoro constraints play a fundamental role. Only recently a prescription for higher genus computations has been constructed [2], but it is mainly based on geometrical properties and counting of degrees of freedom. The role of the Virasoro constraints is not manifest. (The same type of problem can be also found in the formulation of higher genus prescriptions for topological strings, see for example [24, 25].) The formulation of tree level and higher genus computations for the new models is therefore very interesting.

There is another aspect worth mentioning here. As we have seen the from the computation of the cohomology, the massless states of these models are described by off-shell multiplets of super Yang-Mills. This means that some additional constraints should be added in order to describe the physical massless and massive degrees of freedom. These additional constraints play an important role in the formulation of amplitudes because the auxiliary fields and gauge degrees of freedom should drop out from correlation functions. Since one of the main differences between the new lower-dimensional models and the pure-spinor superstring in $d = 10$ is that the supersymmetric multiplets are off-shell when $d < 10$, we hope that from the models presented in this paper the Virasoro constraints can be understood.

Note added: after this paper was completed, we received a copy of [26] where pure-spinor superstrings in $d = 4$ are also discussed. In particular, the role of the BRST operator one obtains by explicitly solving the pure-spinor constraint in terms of a Weyl spinor (the operator that we called Q' in section 5) was clarified and a scattering amplitude prescription for the corresponding theory was given. Furthermore, it was argued that the resulting model is related to a chiral sector of superstring theory compactified to $d = 4$ on a Calabi-Yau manifold.

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A. Flat superspace in $d = 4, 6$

A.1 $\mathcal{N} = 1_4$ in $d = 4$

The $\mathcal{N} = 1$, $d = 4$ superspace coordinates are in Dirac notation (x^m, θ^α) $(\alpha = 1, \ldots, 4)$ and in Weyl notation $\left(x^m, \theta^a, \theta^{\dot{a}}\right)$ $(a = 1, 2,\ \dot{a} = 1, 2)$. We denote the 4×4 gamma matrices acting on Dirac spinors by Γ^m. The charge conjugation matrix used to raise and lower indexes will not be written explicitly.

The supersymmetry transformations acting on superfields are generated by (in Dirac notation)

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial\theta^\alpha} - \frac{1}{2}\left(\Gamma^m\theta\right)_\alpha \frac{\partial}{\partial x^m}\,, \tag{A.1}$$

satisfying

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = -\Gamma^m_{\alpha\beta}\frac{\partial}{\partial x^m}\,. \tag{A.2}$$

The vector field $\partial/\partial x^m$ is invariant under the supersymmetry transformations, as is the usual supersymmetric derivative,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\left(\Gamma^m\theta\right)_\alpha \frac{\partial}{\partial x^m}\,, \tag{A.3}$$

which satisfies,

$$\{D_\alpha, D_\beta\} = \Gamma^m_{\alpha\beta}\frac{\partial}{\partial x^m}\,. \tag{A.4}$$

A.2 $\mathcal{N} = (1,0)_8$ in $d = 6$

The $\mathcal{N} = (1,0)$, $d = 6$ superspace coordinates are (x^m, θ^α_I) ($\alpha = 1,\ldots,4$, $I = 1,2$). The supersymmetry transformations acting on superfields are generated by

$$\mathcal{Q}^I_\alpha = \frac{\partial}{\partial\theta^\alpha_I} - \frac{1}{2}\epsilon^{IJ}\left(\gamma^m\theta_J\right)_\alpha \frac{\partial}{\partial x^m}\,, \tag{A.5}$$

satisfying

$$\left\{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\right\} = -\epsilon^{IJ}\gamma^m_{\alpha\beta}\frac{\partial}{\partial x^m}\,. \tag{A.6}$$

The vector field $\partial/\partial x^m$ is invariant under the supersymmetry transformations, as is the usual supersymmetric derivative,

$$D^I_\alpha = \frac{\partial}{\partial\theta^\alpha_I} + \frac{1}{2}\epsilon^{IJ}\left(\gamma^m\theta_J\right)_\alpha \frac{\partial}{\partial x^m}\,, \tag{A.7}$$

which satisfies,

$$\left\{D^I_\alpha, D^J_\beta\right\} = \epsilon^{IJ}\gamma^m_{\alpha\beta}\frac{\partial}{\partial x^m}\,. \tag{A.8}$$

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